Improving Escalier Step Function Approximation with Exact Root-Finding

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Introduction

Background: Step function approximation represents complex continuous functions with simpler piecewise constant functions. The escalier algorithm finds optimal breakpoint positions k_1, k_2, ..., $k_{\square}$ to maximize the explained sum of squares (ESS).

Problem: Traditional grid search methods test discrete positions at fixed intervals (e.g., 0.001 spacing), which can miss optimal solutions, especially for oscillatory functions where breakpoints may fall between grid points.

Objectives

- Replace brute-force grid search with continuous optimization
- Eliminate discretization error in breakpoint selection
- Improve accuracy for oscillatory test functions
- Maintain or improve computational efficiency

Method

Multi-Start Optimization for k_1:

- Run 10 parallel optimization searches from evenly-spaced starting regions
- Use MATLAB's fminbnd for continuous optimization (precision: 10^{-6} vs. 0.001 grid)
- Select global best across all starting points

Exact Root-Finding for k_2, k_3, k_4:

- Solve criticality equations using Chebfun (MATLAB package):
  - $f(k_0,k_1) + f(k_1,k_2) - 2f(k_1) = 0$
- Find exact roots (no discretization error)

Test Setup

- Benchmark Functions:
  - f_1: x^2 (smooth polynomial)
  - f_3: $\log(x)$ (smooth with endpoint behavior)
  - f_6: Levy function (highly oscillatory)
  - f_7: $x \cdot \sin(9x^2/4)$ (oscillatory)
- Parameters:
  - 4 breakpoints, interval [0,2], tolerance 0.001
  - Baseline: Grid search (0.001 spacing)
  - Optimized: Multi-start fminbnd (10^{-6} precision)

Results

Performance Summary Table

Function	Grid ESS	Optimized ESS	Improvement	Speedup
f_1 (x^2)	6.304	6.304	0.0004%	26×
f_3 (log x)	2.042	2.042	0.0003%	3×
f_6 (Levy)	43.618	45.397	4.08%	2×
f_7 (oscillatory)	1.278	1.284	0.47%	39×

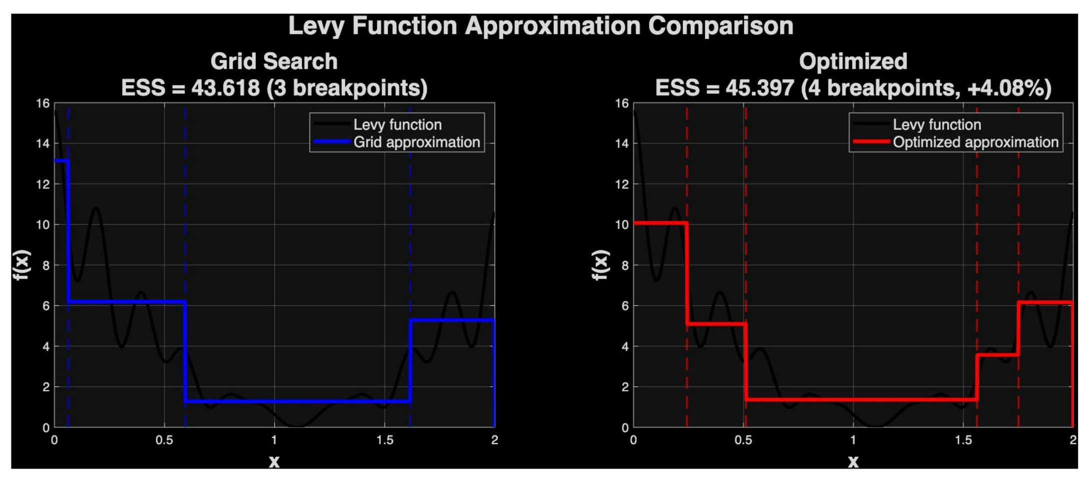
Smooth Functions (f_1, f_3)

The optimized method achieved essentially identical accuracy to grid search (differences < 0.001%) but with significant speedups: 26× faster for f_1 and 3× faster for f_3.

Oscillatory Functions (f_6, f_7)

Function f_6 (Levy) - Key Finding:

The visualization below shows why the optimized method achieves 4.08% better ESS. Grid search (left) uses only 3 breakpoints and misses the oscillatory structure, while the optimized method (right) uses all 4 breakpoints strategically positioned to capture the function's peaks and valleys.



Comparison of grid search vs. optimized approximations for the Levy function.

- Grid: k = [0.063, 0.594, 1.616] (3 breakpoints), ESS = 43.618

- Optimized: k = [0.243, 0.511, 1.562, 1.752] (4 breakpoints), ESS = 45.397

- The first breakpoint at $k_1 \approx 0.243$ captures the initial rise that grid search missed

Function f_7: 0.47% accuracy improvement with 39× speedup (134.7s → 3.5s). Multi-start found the optimal region at $k_1 \approx 0.544$ with higher precision than grid search.

Conclusions

For oscillatory functions:

0.5-4% accuracy improvement by finding breakpoints between grid positions

For smooth functions:

Similar accuracy but 3-26× faster

Multi-start is critical:

Avoids local optima and reliably finds global optimum

Impact:

The 4% improvement on Levy demonstrates measurable value for complex approximation problems

Future Work:

- Test on additional function classes (exponential, rational, Runge)
- Extend to higher-dimensional problems
- Investigate adaptive breakpoint selection
- Explore hybrid approaches combining grid and continuous methods

References

Bossu, S. (2024). Escalier step function approximation. arXiv:2510.16148.